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# Interference and entanglement: an intrinsic approach

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## Abstract

An addition rule of impure density operators, which provides a pure state density operator, is formulated. Quantum interference including a visibility property is discussed in the context of the density operator formalism. A measure of entanglement is then introduced as the norm of the matrix equal to the difference between a bipartite density matrix and the tensor product of partial traces. Entanglement for arbitrary quantum observables for multipartite systems is discussed. Star-product kernels are used to map the formulation of the addition rule of density operators onto the addition rule of symbols of the operators. Entanglement and nonlocalization of the pure state projector and allied operators are discussed. Tomographic and Weyl symbols (tomograms and Wigner functions) are considered as examples. The squeezed states and some spin states (two qubits) are studied to illustrate the formalism.

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## 1. Introduction

The superposition principle of quantum states plays a key role in physical phenomena such as interference of matter waves [1]. Wave properties of electron are connected with the de Broglie wavelength expressed in terms of particle momentum [2]. These properties are naturally described by a wavefunction associated with the particle's quantum state and obeying the Schrödinger equation [3]. For a system with several degrees of freedom, the

possibility of considering two subsystems—the first one connected with some degrees of freedom and the second with the rest of the degrees of freedom, respectively, the superposition principle provides a construction of entangled states [4]. Discussing two subsystems of a given system implies the physical possibility of measuring characteristic properties distinguishing the subsystems.

Entangled states are those which are constructed as a superposition of states, each of which has the wavefunction expressed as a product of wavefunctions depending on different degrees of freedom. The mixed states of quantum systems are described by the density operator [5]. The superposition principle of pure quantum states has been formulated in [6–8] in terms of a new addition rule of the density operators. This addition rule corresponds to a purification procedure of a mixed quantum state obtained by the standard addition rule of the density operators. The relation of the purification procedure to reconstructing the entanglement structure of the mixed state of a bipartite system has been preliminarily discussed in [8].

Various notions of measure of entanglement were suggested in [9–13]. All these measures are related to some operators associated with a bipartite quantum system. The aim of our work is to give the new addition rule of density operators describing the superposition of impure density matrices, and this analysis generalizes the results of [6–8] where the coherent addition rule of pure density operators was formulated. We also define the measure of entanglement of bipartite and multipartite quantum systems considering intrinsic properties of the density operator describing the state.

Since the density operators can be considered using different representations for their symbols, such as Wigner function [14], Husimi–Kano function [15, 16] as well as singular quasidistribution [17, 18], we discuss the addition rule of the density operators in terms of the addition rule of their symbols. To do this, we discuss the star-product of the operator symbols (see, for example, [19–23]). We also consider the case of density operator representation by a standard tomographic probability distribution, which is used to give a ‘probability’ formulation of quantum mechanics [24]. We also give the formulation of both aspects of the superposition principle, namely, the coherent addition of impure density operators and measure of entanglement in terms of the star-product quantization procedure.

The paper is organized as follows. In section 2, the basic ideas of the construction of the purification procedure and measure of entanglement are described. In section 3, the review of the addition rule of pure density operators is presented including the presence of a visibility parameter. In section 4, a new formula for the purification of the sum of impure density operators is obtained. In section 5, a short review of the star-product formalism is given. The purification formula for a symbol for an arbitrary kind of density operator which is obtained by the purification of the sum of the symbols of impure density operators is obtained in section 6. The example of the Wigner function addition rule in terms of the star-product kernel is presented in section 7. The addition rule of tomographic symbols is considered in section 8. In section 9, the notions of entanglement and measure of entanglement are discussed in terms of intrinsic properties of the density operator of a composite system, while in section 10 a notion of entanglement is introduced for other observables. The example of two qubits is considered in section 11. A measure of entanglement of multimode squeezed states is presented in section 12. A purification procedure for the separable density matrix is discussed in section 13. In section 14, the role of the fiducial projector used to formulate the superposition principle in terms of density operators is considered. In the conclusions (section 15) some perspectives are discussed, while in the appendix we give the proof of the theorem that for pure state  $\rho_{AB}$  of the bipartite system  $AB$ , the eigenvalues and ranks of reduced density operators  $\rho_A$  and  $\rho_B$  are equal.

## 2. General ideas

The notion of dynamical variables generating an algebra of observables and the notion of states, which are dual to this algebra, are common to both classical dynamics and quantum dynamics; but in quantum dynamics the operators generating transformations form a noncommutative vector space while the classical algebra of dynamical variables is commutative. Since every true representation (realization) of a commutative algebra is one dimensional, this is no longer true for a noncommutative algebra. This has the immediate consequence that while one can have states in which all dynamical variables have unique values in classical dynamics, this is not so in quantum dynamics. We have fundamental states in quantum dynamics in which all but a (commuting) set of variables have definite (dispersion-free) values for which other dynamical variables have a distribution of values. In the case where the eigenstates are continuous, this finds expression in generalized uncertainty relations [25–28]. The generalized uncertainty relations exist also for discrete observables such as spin (see, e.g., [28]).

Classical states may be associated with distributions in phase space (that is, they have distributions in values for all dynamical variables) of which the extremal (pure) states are points in phase space (their distributions correspond to Dirac delta functions or Kronecker delta matrices for the fundamental dynamical variables). In contrast, the quantum states are linear functionals on the dynamical variables which map nonnegative operators into nonnegative numbers. A linear distribution that can be expressed as a convex combination (sum or integral) of other appropriate ones can be made up of extremal linear functionals which cannot be so decomposed. These extremal states are the ‘pure states’ of a quantum system.

If all quantum states had distributions for some set of dynamical variables, how can we combine them to get dispersion-free states for these dynamical variables? This depends on the characteristic quantum property of superposition of states. Thus, for example, in interference we compose two pure states to form a new pure state. In terms of states considered as linear functionals, the natural process is to generate convex linear sums which are not pure states. But processes such as interference, diffraction and composition of light polarizations need a new procedure.

An alternate formalism suitable for incorporating this composition law is given by the vector (Hilbert) space formalism—with each pure state we associate a vector (and its dual vector), and the expectation values are obtained by the dynamical variables acting on the state as linear operators, and forming the scalar product with the original dual. Real classical variables have their counterparts in self-adjoint linear operators and their expectation value in any state is real. Moreover, positive operators (nonnegative operators) have nonnegative expectation values. Since vectors permit linear combinations (over the field of complex numbers), superposition of states and interference are naturally explained. Since the sesquilinear tensor product of the vector and its dual is a linear functional, this can be compared with the formalism in terms of linear functionals. But the only linear functionals obtained in a sesquilinear form (outer product of a vector and its dual) are extremal. To get a full correspondence with the linear functional formulation, we should form convex combinations of such outer products. Density matrices (nonextremal states) are convex combinations of bilinears in the vectors.

Since the sesquilinear forms (outer products) of states constitute pure states, we see that every ‘state vector’ corresponds to an extremal linear functional, which may be treated as a linear operator in the vector space, we should identify mixed states as nonextremal states of the convex set of states. But the correspondence of a state vector and its pure density operator is not one-to-one. The von Neumann ‘measurement’ is a projector to an eigenstate of a particular self-adjoint dynamical variable. (In the case the spectrum is continuous, von Neumann

prescribes a nested family of spectral projections.) The prescription may be viewed as imparting a measure on the spectrum of the operator representing the dynamical variable.

A generalization of this protocol is called POVM (positive operator-valued measure). A von Neumann measurement results in a pure state of the system (or more generally a density matrix in the eigenspace of the operator). But a POVM may result in a mixed density matrix. The entire set of vectors  $\{\psi e^{i\theta}\}$  corresponds to a unique extremal state

$$\rho = \psi \psi^\dagger.$$

This extremal operator is idempotent, self-adjoint (of trace class) and satisfies

$$\rho^2 = \psi \psi^\dagger \psi \psi^\dagger = \psi \psi^\dagger = \rho$$

since  $\psi^\dagger \psi = 1$  for normalization. This equivalence class  $\{e^{i\theta} \psi\}$  is identified as a ray—so extremal density matrices correspond one-to-one to rays in the vector (Hilbert) space. Rays do not constitute a vector space. Considered as a linear operator, since  $\rho^2 = \rho$  and  $\text{Tr} \rho = \psi^\dagger \psi = 1$ ,  $\rho$  is a projector of rank 1. So extremal states are the projectors associated with unique rays. A mixed state is a probabilistic (nonnegative normalized linear) sum of projectors corresponding to the sum of definite rays with definite probabilities, which may be chosen to be mutually orthogonal.

In forming a superposition of two vectors, their relative phases are important. So the rays by themselves are insufficient. They suffice to form mixtures. The question arises as to how to work with the projectors and yet get a superposition with definite (relative) phases within the formalism of density operators, either pure or mixed.

This is accomplished using the method of ‘purification’ of a nonextremal state. Since there are many possible superpositions, we must have purification of an impure density to any of those states. This is accomplished by using a suitable fiducial projector; the choice of this projector determines the phases (or rather phase differences). We show that there are choices of the fiducial projector that can give any of the superpositions and that, even with one such superposition, the fiducial projector can be chosen amongst a continuous set of projectors.

Having accomplished ‘purification’ we turn to another characteristic property of the quantum system, namely, ‘quantum entanglement’. If we have a composite system, say  $AB$ , composed of two subsystems  $A$  and  $B$ , the generic states of  $AB$  contain information that is not obtained by considering the states of the subsystems. These may be referred to as nonlocal correlations between the subsystems  $A$  and  $B$ , which cannot be attributed to causal connections. (We have processes which have a significant relationship that cannot be accounted for causally; somewhat like the notion ‘synchronicity’ by Jung [29].) Schrödinger pointed out this as a characteristic property representing a quantum system.

We can have correlations between dynamical variables measured in subsystems in classical dynamics. In a pure classical state, this automatically gives pure states with definite values for the subsystem variables. But in the quantum system the situation is entirely different. A general pure state of the composite system gives impure states of the subsystems. In this case, the systems  $A$  and  $B$  are ‘entangled’. For example, the singlet state of two spin-1/2 (particles) is entangled. Any spin component of the combined system  $AB$  gives a zero expectation value (the singlet is total spin-0), the individual particles are completely unpolarized—any spin component has zero expectation value. Yet there is a definite correlation—if one spin is ‘down’, the other is ‘up’ (and vice versa) with respect to any direction.

We recognize that if an entangled pure state is considered as a state of two subsystems  $A$  and  $B$ , they could be mixed with the same rank but with corresponding eigenvalues and eigenprojectors. So the problem of recovering the original pure state from the two mixed (impure) states of the subsystems involves the restoration of nonlocal phase relations, characteristics of entangled states. We have evolved a method of restoring the pure state

using an entangled fiducial projector. However, in this case, the restoration can be done if the subsystems have density matrices of the same rank and same eigenvalues.

We may also recognize that operations on an entangled pure state may lead to an unentangled (Kronecker product) state for the composite system which remains pure when restricted to either subsystem. For example, if we act on the singlet state with the operator of the difference of the two spins, we can obtain a triplet state which may be unentangled. The point is that the operator acting as the difference of the spins is itself ‘nonlocal’ in that it acts on both the subsystems together. Similarly, we can take an unentangled pure state such as the  $\pm 1$  states for the total spin-1, and an entangled pure state can be obtained by either acting with some component of the difference of spins on the singlet state, or by a spin operator antisymmetric in the two spins on the entangled triplet state. Here again the operator has the ability to entangle. We shall make use of this property in generating entanglement. The entanglement and the relative phases are implicit in the fiducial projector chosen.

### 3. Pure state addition

In [6–8], the rule to add two pure states determined by their density operators  $\rho_1$  and  $\rho_2$ , which are projectors, was formulated. This rule corresponds to superposition of the state vectors  $\psi_1$  and  $\psi_2$ . This superposition of vectors is a standard tool to describe the quantum phenomenon of interference. The interference pattern is sensitive to the relative phase of the two vectors. To describe the relative phase, a fiducial projector  $P_0$  was used. By means of the fiducial projector, the quantum interference can be described in terms of operators only without using the state vectors. The addition law of two orthogonal states reads (see [6–8] where the addition formula was written in a slightly different form)

$$\rho = p_1\rho_1 + p_2\rho_2 + \frac{(\rho_1 P_0 \rho_2 + \text{h.c.})\sqrt{p_1 p_2}}{\sqrt{\text{Tr}(\rho_1 P_0 \rho_2 P_0)}} \quad (1)$$

where positive probabilities (numbers  $p_1$  and  $p_2$ ) satisfy the normalization relation

$$p_1 + p_2 = 1. \quad (2)$$

One can check that the density operator (1) is a projector, i.e.

$$\rho^2 = \rho \quad \rho^\dagger = \rho \quad \text{Tr} \rho = 1 \quad \rho_i \rho \rho_i = p_i \rho_i \quad (i = 1, 2) \quad (3)$$

if  $\rho_1$ ,  $\rho_2$  and  $P_0$  are projectors.

We consider the case  $\rho_1 \rho_2 = 0$  and  $P_0 \rho_1 \neq 0$  and  $P_0 \rho_2 \neq 0$ .

The composition law (1) can be interpreted as the purification of the impure density operator

$$\rho_{\text{im}} = p_1 \rho_1 + p_2 \rho_2. \quad (4)$$

Nevertheless, the real meaning of the ‘purification’ consists of the statement that for two given orthogonal projectors and a given fiducial one which is not orthogonal to the given projectors, sum (1) is again a projector. Relation (1) can be extended to describe the purification procedure for the impure density operator of the form

$$\rho_{\text{im}} = \sum_k p_k \rho_k \quad (5)$$

where projectors are such that  $\rho_k \rho_m = 0$  ( $k \neq m$ ) and positive probabilities  $p_k$  satisfy the normalization condition

$$\sum_k p_k = 1. \quad (6)$$

Summation in (5) and (6) can be considered as a summation over a finite set of indices  $k$  or over an infinite one for the system with an infinite number of states. The generalization of (1) provides the purified density operator

$$\rho = \sum_{k,j} \sqrt{p_k p_j} \frac{\rho_k P_0 \rho_j}{\sqrt{\text{Tr}(\rho_k P_0 \rho_j P_0)}}. \quad (7)$$

Formula (1) can also be extended to take into account that there is a visibility parameter  $\gamma$  [8], a characteristic of the interference pattern with  $0 \leq \gamma \leq 1$ . Equation (1) is generalized to the form

$$\rho = p_1 \rho_1 + p_2 \rho_2 + \gamma \frac{\sqrt{p_1 p_2} (\rho_1 P_0 \rho_2 + \text{h.c.})}{\sqrt{\text{Tr}(\rho_1 P_0 \rho_2 P_0)}}. \quad (8)$$

For  $\gamma = 1$ , equation (8) reduces to equation (1).

For  $\gamma = 0$ , one has the impure state (4).

Thus, the visibility parameter  $\gamma$  is a characteristic of completeness of the purification procedure of the density operator or of the degree of decoherence of the initial pure superposition state (1). In the case of (1) or (8), we have  $\rho_k \rho \rho_k = p_k \rho_k$  (no sum on  $k$ ).

Let us now consider two density operators of quantum states which are pure nonorthogonal states. In this case, one has a normalization constant and the purification formula is

$$\rho = \left[ p_1 \rho_1 + p_2 \rho_2 + \frac{\sqrt{p_1 p_2} (\rho_1 P_0 \rho_2 + \text{h.c.})}{\sqrt{\text{Tr}(\rho_1 P_0 \rho_2 P_0)}} \right] \mathcal{N}^{-1}. \quad (9)$$

The normalization constant  $\mathcal{N}$  reads

$$\mathcal{N} = 1 + \frac{2\sqrt{p_1 p_2} \text{Re}(\text{Tr}(\rho_1 P_0 \rho_2))}{\sqrt{\text{Tr}(\rho_1 P_0 \rho_2 P_0)}}. \quad (10)$$

#### 4. Impure state addition

We discuss now the addition rule of two mixed states.

In the case of two impure states  $\rho_1$  and  $\rho_2$ , their sum can be decomposed in terms of orthogonal projectors  $R_n$  (i.e. satisfying  $R_n R_m = \delta_{nm} R_n$ ):

$$p_1 \rho_1 + p_2 \rho_2 = \sum_n \omega_n R_n \quad \sum_n \omega_n = 1 \quad \omega_n \geq 0. \quad (11)$$

One can consider formula (11) as giving the result of a mixture of pure states  $R_n$  in (5). Thus the procedure of addition of impure states can be fulfilled as follows. First, one writes the sum of impure states as a convex sum of orthogonal projectors and then carries out the purification given in equation (7).

In this case, one gets the pure density operator for the result of 'deformed' addition of two impure states  $\rho_1$  and  $\rho_2$  with probabilities  $p_1$  and  $p_2$  given by the following expression which is a generalization of equation (11):

$$p_1 \rho_1 \oplus p_2 \rho_2 = \sum_{k,j} \sqrt{\omega_k \omega_j} \frac{R_k P_0 R_j}{\sqrt{\text{Tr}(R_k P_0 R_j P_0)}} \quad (12)$$

where  $R_k$  are the orthogonal eigenprojectors of the density operator and  $\omega_k$  are the nonnegative eigenvalues of the density operator, i.e.

$$(p_1 \rho_1 + p_2 \rho_2) R_k = \omega_k R_k. \quad (13)$$

There is no sum on  $k$ . The fiducial projector  $P_0$  is chosen to satisfy the condition  $P_0 R_k \neq 0$ .

If one has the addition of more than two impure states, i.e. the density operator of the impure state has the form  $\sum_{s=1}^N p_s \rho_s$ , the result of the ‘deformed’ addition rule has the same form as equation (12), namely,

$$\sum_{s=1}^N \oplus p_s \rho_s = \sum_{kj} \sqrt{\omega_k \omega_j} \frac{R_k P_0 R_j}{\sqrt{\text{Tr}(R_k P_0 R_j P_0)}} \tag{14}$$

where the eigenprojectors  $R_k$  and nonnegative eigenvalues  $\omega_k$  satisfy the equation

$$\left( \sum_{s=1}^N p_s \rho_s \right) R_k = \omega_k R_k \quad (\text{no sum on } k). \tag{15}$$

Thus the purification procedure which is expressed by the deformed addition rule denoted by the sign  $\oplus$  on the left-hand side of equations (12) and (14) is reduced to obtaining eigenvectors (eigenprojectors) and eigenvalues of the nonnegative density operator  $\sum_s p_s \rho_s$  and applying the ansatz with the fiducial projector  $P_0$  ( $P_0 R_k \neq 0$ ) to construct the nonlinear expression on the right-hand side of equations (12) and (14). This expression provides the purified density operator.

In the limit case where the initial density operators  $\rho_s$  are orthogonal projectors, equation (14) is reduced to equation (7) with the obvious replacement  $\omega_k \rightarrow p_k$ . One can point out that only one projector  $P_0$  is sufficient to provide  $(N - 1)$  independent phase parameters in the case of addition of  $N$  orthogonal pure states. This projector must have nonzero overlap with all added pure state projectors. In the case of addition of  $N$  impure states, the number of independent phase parameters, which are contained in only one fiducial projector  $P_0$ , depends on the rank of the density operator  $\sum_s p_s \rho_s$  and is one less than the rank. For clarity, we point out that the obtained addition formula for impure density operators assumes the existence of the fiducial projector. This existence is obvious from a geometrical point of view, but the explicit finding of this fiducial projector for given density operators is a different problem.

**5. Symbols and their star-product**

In quantum mechanics, observables are described by linear operators acting on the Hilbert space of states. In order to consider observables as functions, we first review a general construction [22] and provide general relations and properties of a map from operators onto functions having in mind a map of the density operator onto a function such as a Wigner distribution. Given a Hilbert space  $H$  and an operator  $\hat{A}$  acting on this space, let us suppose that we have a set of operators  $\hat{U}(\mathbf{x})$  acting on  $H$ , where an  $n$ -dimensional vector  $\mathbf{x} = (x_1, x_2, \dots, x_n)$  labels the particular operator in the set. We construct the  $c$ -number function  $f_{\hat{A}}(\mathbf{x})$  (we call it the symbol of operator  $\hat{A}$ ) using the definition

$$f_{\hat{A}}(\mathbf{x}) = \text{Tr}[\hat{A} \hat{U}(\mathbf{x})]. \tag{16}$$

Let us suppose that there must exist a set of operators  $\hat{D}(\mathbf{x})$  such that

$$\hat{A} = \int f_{\hat{A}}(\mathbf{x}) \hat{D}(\mathbf{x}) d\mathbf{x}. \tag{17}$$

We will consider relations (16) and (17) as relations determining the invertible map from the operator  $\hat{A}$  onto the function  $f_{\hat{A}}(\mathbf{x})$ .

The product (star-product) of two functions  $f_{\hat{A}}(\mathbf{x})$  and  $f_{\hat{B}}(\mathbf{x})$  corresponding to two operators  $\hat{A}$  and  $\hat{B}$  is defined by the relations

$$f_{\hat{A}\hat{B}}(\mathbf{x}) = f_{\hat{A}}(\mathbf{x}) * f_{\hat{B}}(\mathbf{x}) := \text{Tr}[\hat{A} \hat{B} \hat{U}(\mathbf{x})]. \tag{18}$$



Since the standard product of operators on a Hilbert space is an associative product, formula (18) defines an associative product for the functions  $f_{\hat{A}}(\mathbf{x})$ .

## 6. Superposition rule in terms of symbols of density operators

Using formulae (16) and (17), one can write a composition rule for two symbols  $f_{\hat{A}}(\mathbf{x})$  and  $f_{\hat{B}}(\mathbf{x})$ , which determines the star-product of these symbols bilinear in the two symbols,

$$f_{\hat{A}}(\mathbf{x}) * f_{\hat{B}}(\mathbf{x}) = \int f_{\hat{A}}(\mathbf{x}'') f_{\hat{B}}(\mathbf{x}') K(\mathbf{x}'', \mathbf{x}', \mathbf{x}) d\mathbf{x}' d\mathbf{x}'' \quad (19)$$

The kernel in the integral of (19) is determined by the trace of product of the basic operators, which we use to construct the map

$$K(\mathbf{x}'', \mathbf{x}', \mathbf{x}) = \text{Tr}[\hat{D}(\mathbf{x}'') \hat{D}(\mathbf{x}') \hat{U}(\mathbf{x})]. \quad (20)$$

Formula (20) can be extended for the case of the star-product of  $N$  symbols of operators  $\hat{A}_1, \hat{A}_2, \dots, \hat{A}_N$ . Thus one has

$$f_{\hat{A}_1}(\mathbf{x}) * f_{\hat{A}_2}(\mathbf{x}) * \dots * f_{\hat{A}_N}(\mathbf{x}) = \int f_{\hat{A}_1}(\mathbf{x}_1) f_{\hat{A}_2}(\mathbf{x}_2) \dots f_{\hat{A}_N}(\mathbf{x}_N) K(\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_N, \mathbf{x}) d\mathbf{x}_1 d\mathbf{x}_2 \dots d\mathbf{x}_N \quad (21)$$

where the kernel has the form [22]

$$K(\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_N, \mathbf{x}) = \text{Tr}[\hat{D}(\mathbf{x}_1) \hat{D}(\mathbf{x}_2) \dots \hat{D}(\mathbf{x}_N) \hat{U}(\mathbf{x})]. \quad (22)$$

The trace of an operator  $\hat{A}^N$  is determined by the kernel as follows:

$$\text{Tr} \hat{A}^N = \int f_{\hat{A}}(\mathbf{x}_1) f_{\hat{A}}(\mathbf{x}_2) \dots f_{\hat{A}}(\mathbf{x}_N) \text{Tr}[\hat{D}(\mathbf{x}_1) \hat{D}(\mathbf{x}_2) \dots \hat{D}(\mathbf{x}_N)] d\mathbf{x}_1 d\mathbf{x}_2 \dots d\mathbf{x}_N \quad (23)$$

$$\text{Tr}(\hat{A} \hat{B}) = \int f_{\hat{A}}(\mathbf{x}_1) f_{\hat{B}}(\mathbf{x}_2) \text{Tr}[\hat{D}(\mathbf{x}_1) \hat{D}(\mathbf{x}_2)] d\mathbf{x}_1 d\mathbf{x}_2. \quad (24)$$

When the operator  $\hat{A}$  is a density operator of a quantum state, formula (23) for  $N = 2$  determines the purity parameter of the state.

Formulae (21) and (22) can be used to formulate the addition law of the density operators of orthogonal pure states  $\rho_i$  as an addition law for their symbols  $f_{\rho_i}(\mathbf{x})$ . In the case of purification of the sum  $\sum_k p_k \rho_k$  of pure states  $\rho_k$  by means of a fiducial projector  $P_0$ , one has the following symbols:

$$\begin{aligned} f_{\rho}(\mathbf{x}) & \text{ for the purified density operator,} \\ f_{\rho_k}(\mathbf{x}) & \text{ for the pure state with density operator } \rho_k, \\ f_0(\mathbf{x}) & \text{ for the fiducial projector } P_0. \end{aligned}$$

The formula describing the quantum interference in terms of symbols of the density operators reads

$$f_{\rho}(\mathbf{x}) = \sum_{jk} \sqrt{p_k p_j} \frac{\int f_{\rho_k}(\mathbf{x}_1) f_0(\mathbf{x}_2) f_{\rho_j}(\mathbf{x}_3) K(\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \mathbf{x}) d\mathbf{x}_1 d\mathbf{x}_2 d\mathbf{x}_3}{\sqrt{\int f_{\rho_k}(\mathbf{x}_1) f_0(\mathbf{x}_2) f_{\rho_j}(\mathbf{x}_3) f_0(\mathbf{x}_4) k(\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \mathbf{x}_4) d\mathbf{x}_1 d\mathbf{x}_2 d\mathbf{x}_3 d\mathbf{x}_4}} \quad (25)$$

where the kernel  $K(\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \mathbf{x})$  is defined by equation (22) while the kernel  $k(\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \mathbf{x}_4)$ , which determines the trace of the product of  $N$  operators ( $N = 4$ ) in terms of their symbols, reads

$$k(\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_N) = \text{Tr}[\hat{D}(\mathbf{x}_1) \hat{D}(\mathbf{x}_2) \dots \hat{D}(\mathbf{x}_N)]. \quad (26)$$

In the case of purification of the sum of impure states (14), one has an analogous formula with the replacement  $\rho_k \rightarrow R_k$  and  $p_k \rightarrow \omega_k$ .

## 7. Weyl symbol addition and interference in terms of Wigner–Moyal functions

In this section, we will consider a known example of Wigner distribution which is related to the Heisenberg–Weyl-group representation. The interference can be described in terms of Wigner functions using formula (25). The displacement operator

$$\hat{D}(\alpha_x) = \exp(\alpha_x \hat{a}^\dagger - \alpha_x^* \hat{a}) \quad (27)$$

where

$$\alpha_x = x_1 + ix_2 \quad \alpha_x^* = x_1 - ix_2 \quad (28)$$

and real numbers  $x_1$  and  $x_2$  are expressed in terms of position and momentum as

$$x_1 = \frac{q}{\sqrt{2}} \quad x_2 = \frac{p}{\sqrt{2}} \quad (29)$$

determines the basic operators defining the Weyl map. Thus, one has for the basic operators of the map the following expressions [22]:

$$\hat{U}(\mathbf{x}) = 2\hat{D}(\alpha_x)(-1)^{\hat{a}^\dagger \hat{a}} \hat{D}(-\alpha_x) \quad (30)$$

$$\hat{D}(\mathbf{x}) = \frac{2}{\pi} \hat{D}(\alpha_x)(-1)^{\hat{a}^\dagger \hat{a}} \hat{D}(-\alpha_x). \quad (31)$$

The operator  $(-1)^{\hat{a}^\dagger \hat{a}}$  is the parity operator,  $(-1)^{\hat{a}^\dagger \hat{a}} = P$ , with the matrix elements given in the position (or momentum) representation by the formula

$$\langle x | \hat{P} | y \rangle = \delta(x + y). \quad (32)$$

The Weyl symbol of the density operator  $\rho$  is defined by (16) where we use operator (30) and make the replacement  $f \rightarrow W$ , which is the state Wigner function, and it reads

$$W_\rho(\alpha) = 2 \text{Tr}[\rho \hat{D}(\alpha)(-1)^{\hat{a}^\dagger \hat{a}} \hat{D}(-\alpha)] \quad (\alpha \equiv \alpha_x).$$

To describe the star-product of Weyl symbols, we introduce a generalization of notation (29)

$$\mathbf{x}_k = (x_{k_1}, x_{k_2}) \quad x_{k_1} = \frac{q_k}{\sqrt{2}} \quad x_{k_2} = \frac{p_k}{\sqrt{2}}.$$

Then

$$\alpha_k = \frac{1}{\sqrt{2}}(q_k + ip_k) \quad k = 1, 2, \dots, N.$$

The kernel of the star-product of  $(N - 1)$  Weyl symbols has the form

$$K(\alpha_1, \alpha_2, \dots, \alpha_N) = \text{Tr} \left[ \hat{U}(\mathbf{x}_N) \prod_{k=1}^{N-1} \hat{D}(\mathbf{x}_k) \right]. \quad (33)$$

The kernel can be rewritten in terms of the complex numbers  $\alpha_i$  ( $i = 1, 2, \dots, N$ ) as [22]

$$K(\alpha_1, \alpha_2, \dots, \alpha_N) = \frac{2^{N-1}}{\pi^{N-1}} \exp \left\{ \sum_{j>i}^{N-1} \sum_{i=1}^{N-1} 2(q^{j-i+2-N} \alpha_i \alpha_j^* + q^{i-j} \alpha_j \alpha_i^*) \right. \\ \left. + \sum_{i=1}^{N-1} 2(q^{1-i} \alpha_i \alpha_N^* + q^{i+1-N} \alpha_N \alpha_i^*) \right\} \quad (34)$$

where  $q = -1$ .

The kernel for the trace of the product of four operators reads

$$k(\alpha_1, \alpha_2, \alpha_3, \alpha_4) = \frac{4}{\pi^3} \delta^{(2)}(\alpha_1 - \alpha_2 + \alpha_3 - \alpha_4) \exp\{-2[(\alpha_1\alpha_2^* - \alpha_1\alpha_3^* + \alpha_1\alpha_4^* + \alpha_2\alpha_3^* - \alpha_2\alpha_4^* + \alpha_3\alpha_4^*) - \text{c.c.}]\}. \quad (35)$$

Having the above kernels we can obtain the Weyl symbol of the pure density operator (which we found by means of purification of a mixture of several states) by inserting the kernels and the Wigner functions into (25). The explicit result for the addition of two Wigner functions was given in [6] in a different form.

## 8. Symplectic tomograms and superposition principle

Now we consider the example of tomograms.

According to the general scheme, one can introduce for an operator  $\hat{A}$  the tomographic symbol  $f_{\hat{A}}(\mathbf{x})$ , where  $\mathbf{x} = (x_1, x_2, x_3) \equiv (X, \mu, \nu)$ , which we denote here as  $w_{\hat{A}}(X, \mu, \nu)$  depending on the position  $X$  and the parameters  $\mu$  and  $\nu$  of the reference frame [22],

$$w_{\hat{A}}(X, \mu, \nu) = \text{Tr}[\hat{A}\hat{U}(\mathbf{x})].$$

The operator  $\hat{U}(\mathbf{x})$  is given by

$$\hat{U}(\mathbf{x}) \equiv \hat{U}(X, \mu, \nu) = \delta(X - \mu\hat{q} - \nu\hat{p}) = |X|\delta\left(1 - \frac{\mu\hat{q}}{X} - \frac{\nu\hat{p}}{X}\right)$$

where  $\hat{q}$  and  $\hat{p}$  are the position and momentum operators.

The inverse transform will be of the form

$$\hat{A} = \int w_{\hat{A}}(X, \mu, \nu) \hat{D}(X, \mu, \nu) dX d\mu d\nu$$

where

$$\hat{D}(\mathbf{x}) \equiv \hat{D}(X, \mu, \nu) = \frac{1}{2\pi} \exp(iX - i\nu\hat{p} - i\mu\hat{q}).$$

The kernel defining the star-product of two tomograms

$$K(\mathbf{x}'', \mathbf{x}', \mathbf{x}) = \text{Tr}[\hat{D}(X'', \mu'', \nu'') \hat{D}(X', \mu', \nu') \hat{U}(X, \mu, \nu)]$$

reads [22]

$$K(X_1, \mu_1, \nu_1, X_2, \mu_2, \nu_2, X, \mu, \nu) = \frac{\delta(\mu(\nu_1 + \nu_2) - \nu(\mu_1 + \mu_2))}{4\pi^2} \times \exp\left[\frac{i}{2}\left((\nu_1\mu_2 - \nu_2\mu_1) + 2X_1 + 2X_2 - \frac{2(\nu_1 + \nu_2)}{\nu}X\right)\right].$$

The trace of the product of four basic operators, which provides the kernel to calculate the denominator in the addition formula of density operators (25) reads

$$k(\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_N) = \text{Tr}\left[\prod_{k=1}^N \hat{D}(X_k, \mu_k, \nu_k)\right] = (2\pi)^{1-N} \delta\left(\sum_{k=1}^N \mu_k\right) \delta\left(\sum_{k=1}^N \nu_k\right) \times \exp\left\{i\left(\sum_{k=1}^N X_k + \frac{1}{2} \sum_{k<s=1}^N (\nu_k\mu_s - \mu_k\nu_s)\right)\right\} \quad N = 4. \quad (36)$$

Having the above kernels we can obtain the tomogram of the pure density operator by inserting the kernels into (25) and making there the replacement  $f \rightarrow w$ . In a different form the explicit result for the addition of tomograms was obtained in [6].

## 9. Notion of entanglement

Another quantum-mechanical property related to the superposition principle of states in bipartite and multipartite systems is entanglement. Let us have density operator  $\rho_{AB}$  of composite system  $AB$  which has two subsystems  $A$  and  $B$ . This means that there exist experimental possibilities for measuring the properties of the subsystem  $A$  and subsystem  $B$ . The density operator  $\rho_{AB}$  determines two density operators of the subsystems

$$\rho_A = \text{Tr}_B \rho_{AB} \quad \text{and} \quad \rho_B = \text{Tr}_A \rho_{AB}.$$

Let us consider the tensor product of the two subsystem density operators

$$\rho_{A \times B} = \rho_A \otimes \rho_B.$$

There is a difference in the two density operators

$$R_{AB} = \rho_{AB} - \rho_A \otimes \rho_B.$$

This difference is a characteristic of entanglement. If the system is in the state  $\rho_{AB}$ , which is disentangled, the operator  $R_{AB} = 0$ . The numerical characteristic of entanglement is described by nonzero matrix elements of the operator  $R_{AB}$ . A basic independent (invariant) characteristic of the operator  $R_{AB}$  is the number

$$e = \text{Tr}(R_{AB} R_{AB}^\dagger). \quad (37)$$

This number can be considered as a measure of entanglement. Since  $R_{AB}^\dagger = R_{AB}$  one has

$$e = \text{Tr}(R_{AB}^2).$$

There are other numerical characteristics of entanglement such as traces of higher powers of the matrix  $R_{AB}$

$$e^{(n)} = \text{Tr}(R_{AB}^{n+1}).$$

The state  $\rho_{AB}$  is characterized by the purity parameter

$$\mu_{AB} = \text{Tr} \rho_{AB}^2$$

and the state  $\rho_{A \times B}$  has its own purity parameter

$$\mu_{A \times B} = \mu_A \mu_B$$

where

$$\mu_A = \text{Tr} \rho_A^2 \quad \text{and} \quad \mu_B = \text{Tr} \rho_B^2.$$

Since

$$R_{AB}^2 = \rho_{AB}^2 + (\rho_A \otimes \rho_B)^2 - \rho_{AB} \rho_A \otimes \rho_B - \rho_A \otimes \rho_B \rho_{AB} \quad (38)$$

one has

$$\text{Tr} R_{AB}^2 = \text{Tr} \rho_{AB}^2 + \text{Tr}(\rho_A^2 \otimes \rho_B^2) - 2 \text{Tr}(\rho_{AB} \rho_A \otimes \rho_B). \quad (39)$$

Thus we get the measure of entanglement in the form

$$e = \mu_{AB} + \mu_A \mu_B - 2 \sqrt{\mu_{AB} \mu_A \mu_B} \cos \theta. \quad (40)$$

The last term on the right-hand side of (40) is determined as

$$\sqrt{\mu_{AB} \mu_A \mu_B} \cos \theta = \text{Tr}(\rho_{AB} \rho_A \times \rho_B).$$

For the pure state  $\mu_{AB} = 1$ , one has  $\mu_A = \mu_B = \mu$  (see the appendix) and it gives  $e = 1 + \mu^2 - 2\mu \cos \theta$ . The introduced angle  $\theta$  and parameters  $\mu_A$ ,  $\mu_B$  and  $\mu_{AB}$  can be

functionally dependent. In view of this, finding maxima or minima of the entanglement measure needs to take into account this dependence.

In fact, the measure of entanglement (37) is defined using the notion of distance between two density operators (see, e.g., [30]). The connection of distance with measure of entanglement is natural and it was discussed, e.g., in [31–34]. In the present work, we use the Hilbert–Schmidt distance as a measure of entanglement but the novelty of the suggested measure of entanglement is related to the choice of the density operators being compared. We use the distance between the system density operator and the tensor product of the partial traces over the subsystem degrees of freedom. This characteristic is intrinsic because it is contained in the state density operator only. The geometrical sense of the notion of the entanglement measure can be clarified using an analogy with distance between the points on Euclidean vectors  $|\mathbf{a} - \mathbf{b}|$ , where real vectors  $\mathbf{a}$  and  $\mathbf{b}$  describe the points. With this definition, a partially separable system has a nonzero entanglement.

Each matrix can be considered as a complex vector. The standard scalar product of any two vectors  $\mathbf{C}$  and  $\mathbf{D}$  can always be treated as

$$\mathbf{C} \cdot \mathbf{D} = \sum_s C_s^* D_s.$$

If one considers the matrix elements of the two matrices  $C$  and  $D$  as components of the vectors  $\mathbf{C}$  and  $\mathbf{D}$ , one also has

$$\mathbf{C} \cdot \mathbf{D} = \text{Tr}(DC^\dagger).$$

For Hermitian matrices  $C = C^\dagger$  and  $D = D^\dagger$ ,

$$\mathbf{C} \cdot \mathbf{D} = \text{Tr}(CD).$$

If one considers a density matrix as the vector, the purity parameter plays the role of the square of the vector length, so one has for the purity parameter the inequality

$$0 < \mu \leq 1.$$

The description of the matrix  $R_{AB}$  as a vector makes it obvious that measure of entanglement (37) coincides with the square of the vector length, which in turn is the difference of two other vectors. This means that the length of the vectors under consideration, which correspond to normalized density matrices, is less than unity. Thus, the geometrical interpretation of the measure of entanglement means that angle  $\theta$  in (40) is the angle between the two vectors. This angle can depend on the length of the vectors determining the purity parameters of the system and subsystems. The angle parameter is introduced only in order to illustrate the geometrical picture of the entanglement measure under discussion.

## 10. Entanglement for arbitrary observables

Usually the notion of entanglement is applied for the density operator. Mathematically, the construction of the measure of entanglement  $e$  given by (37) can be extended for arbitrary observable  $\hat{O}$  represented in the form of a sum of projectors,

$$\hat{O} = \sum_n a_n \hat{P}_n$$

where the  $\hat{P}_n$  are the eigenprojectors and  $a_n$  the eigenvalues of the observable  $\hat{O}$ , i.e.

$$\hat{O} \hat{P}_n = a_n \hat{P}_n.$$

For a density operator, the eigenvalues are nonnegative numbers. For an arbitrary observable, the eigenvalues are real numbers and they can take negative values. If one has a prescribed

division of the system in terms of two subsystems  $A$  and  $B$ , the observable  $\hat{O}$  which acts in the Hilbert space of the system can be treated in the same manner as the density operator in the previous section. Thus one can define the reduced observables

$$\hat{O}_A = \text{Tr}_B \hat{O} \quad \hat{O}_B = \text{Tr}_A \hat{O}.$$

The tensor product of the observables

$$\hat{O}_{A \times B} = \hat{O}_A \otimes \hat{O}_B$$

acts in the Hilbert space of the system.

The correlations of two subsystems captured by the observable  $\hat{O}$  can be connected with a measure of entanglement as in the case of the density operator. We define the measure of entanglement for the observable  $\hat{O}$  as the number

$$e_0 = \text{Tr}[(\hat{O} - \hat{O}_A \otimes \hat{O}_B)^2] = \text{Tr}[(\hat{O} - \hat{O}_A \otimes \hat{O}_B)(\hat{O}^\dagger - \hat{O}_A^\dagger \otimes \hat{O}_B^\dagger)].$$

This number gives the invariant description of a ‘distance’ between two Hermitian operators  $\hat{O}$  and  $\hat{O}_A \otimes \hat{O}_B$  in exactly the same manner as in the case of the distance between two density operators.

Analogously, one can introduce positive parameters

$$\mu_0 = \sum_n a_n^2 \quad \mu_{0A} = \sum_k a_{kA}^2 \quad \mu_{0B} = \sum_\alpha a_{\alpha B}^2$$

where  $a_{kA}$  and  $a_{\alpha B}$  are the eigenvalues of the Hermitian matrices  $\hat{O}_A$  and  $\hat{O}_B$ , respectively. So, formula (40) can be extended for an arbitrary observable in the form

$$e_0 = \mu_0 + \mu_{0A}\mu_{0B} - 2\sqrt{\mu_0\mu_{0A}\mu_{0B}} \cos \theta$$

where we define  $\cos \theta$  using the same geometrical interpretation of the scalar product of two vectors

$$\sqrt{\mu_0\mu_{0A}\mu_{0B}} \cos \theta = \text{Tr}(\hat{O} \hat{O}_A \otimes \hat{O}_B).$$

Thus we introduced the notion of entanglement for other Hermitian observables than density operators. Of course, the inequalities for purity parameters in the case of density operators are not valid for other observables.

One can make a generalization introducing the measure of entanglement of the  $k$ th order of an arbitrary observable to multipartite system  $AB \cdots C$  using the definition of measure

$$e_0^{(k)} = \text{Tr}[(\hat{O} - \hat{O}_A \otimes \hat{O}_B \otimes \cdots \otimes \hat{O}_C)^k] \quad \hat{O}_A = \text{Tr}_{B \cdots C} \hat{O}, \dots \quad k = 2, 3, \dots, N. \tag{41}$$

For even  $k$ , the above parameter is a nonnegative number. The measure can be normalized using the factor  $\mu_0^{-1}$ .

### 11. Example of two qubits

Let us consider a density matrix with unit trace for two spins in the basis  $|\uparrow\rangle$  and  $|\downarrow\rangle$  for the first and second spins, correspondingly, i.e., in the basis in four-dimensional space

$$|\uparrow\uparrow\rangle \quad |\uparrow\downarrow\rangle \quad |\downarrow\uparrow\rangle \quad |\downarrow\downarrow\rangle.$$

The Hermitian density matrix has a form

$$\rho = \begin{pmatrix} \rho_{11} & \rho_{12} & \rho_{13} & \rho_{14} \\ \rho_{21} & \rho_{22} & \rho_{23} & \rho_{24} \\ \rho_{31} & \rho_{32} & \rho_{33} & \rho_{34} \\ \rho_{41} & \rho_{42} & \rho_{43} & \rho_{44} \end{pmatrix}. \tag{42}$$

The density matrix  $\rho_A = \text{Tr}_B \rho$  reads

$$\rho_A = \begin{pmatrix} \rho_{11} + \rho_{22} & \rho_{13} + \rho_{24} \\ \rho_{31} + \rho_{42} & \rho_{33} + \rho_{44} \end{pmatrix} \quad (43)$$

and the density matrix  $\rho_B = \text{Tr}_A \rho$  reads

$$\rho_B = \begin{pmatrix} \rho_{11} + \rho_{33} & \rho_{12} + \rho_{34} \\ \rho_{21} + \rho_{43} & \rho_{22} + \rho_{44} \end{pmatrix}. \quad (44)$$

The tensor product of two matrices  $\rho_A$  and  $\rho_B$  has the form quadratic in matrix elements of the matrix  $\rho$

$$\rho_A \otimes \rho_B = \begin{vmatrix} (\rho_{11} + \rho_{22})\rho_B & (\rho_{13} + \rho_{24})\rho_B \\ (\rho_{31} + \rho_{42})\rho_B & (\rho_{33} + \rho_{44})\rho_B \end{vmatrix}.$$

The purity parameter of the two-spin state (42) equals

$$\mu = \sum_{i,k=1}^4 |\rho_{ik}|^2. \quad (45)$$

The purity parameters of the states (43) and (44) read

$$\mu_A = |\rho_{11} + \rho_{22}|^2 + |\rho_{13} + \rho_{24}|^2 + |\rho_{31} + \rho_{42}|^2 + |\rho_{33} + \rho_{44}|^2 \quad (46)$$

and

$$\mu_B = |\rho_{11} + \rho_{33}|^2 + |\rho_{12} + \rho_{34}|^2 + |\rho_{21} + \rho_{43}|^2 + |\rho_{22} + \rho_{44}|^2. \quad (47)$$

One can calculate the trace defining the angle between two vectors corresponding to the density operators in the form

$$\begin{aligned} \text{Tr}(\rho \rho_A \otimes \rho_B) &= (\rho_{11} + \rho_{22})[\rho_{11}(\rho_{11} + \rho_{33}) + \rho_{12}(\rho_{21} + \rho_{43}) + \rho_{21}(\rho_{12} + \rho_{34}) + \rho_{22}(\rho_{22} + \rho_{44})] \\ &+ (\rho_{31} + \rho_{42})[\rho_{13}(\rho_{11} + \rho_{33}) + \rho_{14}(\rho_{21} + \rho_{43}) + \rho_{23}(\rho_{12} + \rho_{34}) \\ &+ \rho_{24}(\rho_{22} + \rho_{44})] + (\rho_{13} + \rho_{24})[\rho_{31}(\rho_{11} + \rho_{33}) + \rho_{32}(\rho_{21} + \rho_{43}) + \rho_{41}(\rho_{12} + \rho_{34}) \\ &+ \rho_{42}(\rho_{22} + \rho_{44})] + (\rho_{33} + \rho_{44})[\rho_{33}(\rho_{11} + \rho_{33}) + \rho_{34}(\rho_{21} + \rho_{43}) \\ &+ \rho_{43}(\rho_{12} + \rho_{34}) + \rho_{44}(\rho_{22} + \rho_{44})]. \end{aligned} \quad (48)$$

Having expressions (40), (45)–(48) one can calculate the measure of entanglement for an arbitrary density matrix of two spins. For example, in the case of a pure state

$$\rho = \frac{1}{2} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad (49)$$

one has

$$\mu = 1 \quad \mu_A = \frac{1}{2} \quad \mu_B = \frac{1}{2}. \quad (50)$$

This provides the maximum entanglement of the state (49), i.e.

$$e = \frac{3}{4}. \quad (51)$$

For a more general matrix of pure state of the form

$$\rho = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & c^2 & sc & 0 \\ 0 & sc & s^2 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad (52)$$

where  $c \equiv \cos \varphi$  and  $s \equiv \sin \varphi$ , one gets

$$e_\varphi = \frac{1}{2} \sin^2 2\varphi \left[ 1 + \frac{\sin^2 2\varphi}{2} \right]. \quad (53)$$

The angle in (40) is determined by the angle  $\varphi$  of (53). For  $\varphi = 45^\circ$ , one has  $\theta = 60^\circ$ .

## 12. Distance and entanglement of Gaussian squeezed states

One can use the developed approach to study entanglement of two-mode squeezed Gaussian states. The Wigner function of generic squeezed and correlated state  $\rho$  in  $n$  dimensions has the form ( $\hbar = 1$ ) [28]

$$W(\mathbf{Q}) = N \exp\left[-\frac{1}{2}(\underline{\mathbf{Q}}) \Sigma^{-1}(\underline{\mathbf{Q}})\right] \quad N = (\det \Sigma)^{-1/2} \quad (54)$$

where  $\mathbf{Q} = (p_1, p_2, \dots, p_n, q_1, q_2, \dots, q_n)$  and

$$\underline{\mathbf{Q}} = \mathbf{Q} - \langle \mathbf{Q} \rangle \quad (55)$$

with  $\langle \mathbf{Q} \rangle$  being the parameters describing means of the quadrature components. The  $2n \times 2n$  matrix  $\Sigma$  describes variances and covariances of the quadrature components. We present the dispersion matrix in the form

$$\Sigma = \begin{pmatrix} \Sigma_1 & \Sigma_{12} \\ \Sigma'_{12} & \Sigma_2 \end{pmatrix} \quad \Sigma^{-1} = \begin{pmatrix} A & B \\ B' & C \end{pmatrix}. \quad (56)$$

We consider two subsystems with dimensions  $n_1$  and  $n_2$ , with  $n_1 + n_2 = n$ . Let us suppose that the system state has the parameters  $\langle \mathbf{Q} \rangle = 0$  (squeezed vacuum in the case of pure states). The normalization constant  $N$  in (54) is determined by the matrix  $\Sigma$  due to the condition

$$\int W(\mathbf{Q}) \frac{d\mathbf{Q}}{(2\pi)^n} = 1. \quad (57)$$

The purity parameter of the Gaussian state of the system equals [28]

$$\mu = \text{Tr } \rho^2 = \int W^2(\mathbf{Q}) \frac{d\mathbf{Q}}{(2\pi)^n} = 2^{-n} (\det \Sigma)^{-1/2}. \quad (58)$$

Integrals (57) and (58) can be calculated using the formula for an  $n$ -dimensional Gaussian integral

$$\int e^{-\mathbf{x}a\mathbf{x}+b\mathbf{x}} d\mathbf{x} = \frac{\pi^{n/2}}{\sqrt{\det a}} \exp\left(\frac{1}{4}b a^{-1} b\right). \quad (59)$$

The Wigner function of the subsystem state 1  $\rho_1$ , which is denoted as  $W_1(\mathbf{Q}_1)$ , is given by the relation

$$W_1(\mathbf{Q}_1) = \int W(\mathbf{Q}) \frac{d\mathbf{Q}_2}{(2\pi)^{n_2}} \quad (60)$$

and the Wigner function of the subsystem state 2  $\rho_2$  is given by the analogous relation

$$W_2(\mathbf{Q}_2) = \int W(\mathbf{Q}) \frac{d\mathbf{Q}_1}{(2\pi)^{n_1}}. \quad (61)$$

Both integrals are Gaussian. Due to this, one has

$$W_1(\mathbf{Q}_1) = N_1 \exp\left[-\frac{1}{2}(\mathbf{Q}_1 \sigma_1^{-1} \mathbf{Q}_1)\right] \quad N_1 = (\det \sigma_1)^{-1/2} \quad (62)$$

$$W_2(\mathbf{Q}_2) = N_2 \exp\left[-\frac{1}{2}(\mathbf{Q}_2 \sigma_2^{-1} \mathbf{Q}_2)\right] \quad N_2 = (\det \sigma_2)^{-1/2} \quad (63)$$

where

$$\sigma_1^{-1} = A - B C^{-1} B' \quad \sigma_2^{-1} = C - B' A^{-1} B.$$



The purity parameters of the states of the subsystem read

$$\mu_1 = 2^{-n_1} (\det \sigma_1)^{-1/2} \quad \mu_2 = 2^{-n_2} (\det \sigma_2)^{-1/2}. \quad (64)$$

In the case  $\det \Sigma = (1/4)^n$ ,  $\mu_1 = \mu_2$ .

The normalization constants  $N_1$  and  $N_2$  are functions of initial dispersion matrix  $\Sigma$ . The Wigner function of the state  $\rho_1 \otimes \rho_2$  has the product form

$$W_{12}(\mathbf{Q}) = W_1(\mathbf{Q}_1)W_2(\mathbf{Q}_2). \quad (65)$$

This form is also Gaussian

$$W_{12}(\mathbf{Q}) = N_{12} \exp \left[ -\frac{1}{2} (\mathbf{Q} \sigma^{-1} \mathbf{Q}) \right] \quad N_{12} = N_1 N_2 \quad \sigma = \begin{pmatrix} \sigma_1 & 0 \\ 0 & \sigma_2 \end{pmatrix}. \quad (66)$$

To calculate the introduced measure of entanglement, one has to calculate the fidelity  $t = \text{Tr}(\rho \rho_1 \otimes \rho_2)$ , which is expressed in terms of Wigner functions by the integral

$$t = \int \frac{d\mathbf{Q}}{(2\pi)^n} W(\mathbf{Q}) W_{12}(\mathbf{Q}). \quad (67)$$

The integral is Gaussian again with the dispersion parameters  $(\sigma + \Sigma)$ . So one has

$$t = (\det(\Sigma + \sigma))^{-1/2}.$$

This trace determines the term with  $\cos \theta$  in the expression for entanglement of the squeezed Gaussian state.

Thus, the measure of entanglement of the squeezed Gaussian state reads

$$e_G = 2^{-n} (\det \Sigma)^{-1/2} + 2^{-n} (\det \sigma)^{-1/2} - 2(\det(\Sigma + \sigma))^{-1/2}.$$

It is determined by the quadrature dispersion matrix of the composite system, which is characteristic of Gaussians. If the pure state is squeezed but not correlated [26, 27], entanglement is absent. For entanglement, one needs the correlation of quadratures in the initial pure states<sup>4</sup>.

### 13. Purification of separable density matrix

In this section, we consider the procedure of purification of a mixed density matrix. The density matrix of a composite system is said to be simply separable if it has the form

$$\rho_{AB} = \rho_A \otimes \rho_B \quad (68)$$

where  $\rho_A = \text{Tr}_B \rho_{AB}$  and  $\rho_B = \text{Tr}_A \rho_{AB}$ . Such a matrix can be pure if and only if  $\rho_A$  and  $\rho_B$  are pure and hence  $\rho_{AB}$ ,  $\rho_A$  and  $\rho_B$  are projectors in the appropriate spaces. Such a pure state of the composite system is not entangled. More generally, a density matrix  $\rho_{AB}$  is said to be separable if

$$\rho_{AB} = \sum_n p_n \rho_{nA} \otimes \rho_{nB} \quad \sum_n p_n = 1 \quad p_n \geq 0. \quad (69)$$

In this case,

$$\rho_A = \sum_n p_n \rho_{nA} \quad \text{and} \quad \rho_B = \sum_n p_n \rho_{nB}.$$

<sup>4</sup> It is worth noting that another measure of entanglement based on the cross covariances of quadrature components of entangled modes was introduced in [35].

Clearly if  $n \geq 2$ , the density matrix  $\rho_{AB}$  is not pure. For  $n = 1$  the matrix  $\rho_{AB}$  is not pure unless  $\rho_A$  and  $\rho_B$  are one-dimensional projectors. Since  $\rho_{nA}$  and  $\rho_{nB}$  are density matrices, they are convex linear sums of projectors

$$\rho_{nA} = \sum_j p_{nj} \Pi_{nj}^A \quad \text{and} \quad \rho_{nB} = \sum_k q_{nk} \Pi_{nk}^B \quad (70)$$

where  $p_{nj}$  and  $q_{nk}$  are nonnegative numbers,  $\sum_j p_{nj} = \sum_k q_{nk} = 1$ . Thus  $\rho_{AB}$  is the convex sum of projectors

$$\rho_{AB} = \sum_{n,j,k} p_{nj} q_{nk} u_{nj} v_{nk} u_{nj}^\dagger v_{nk}^\dagger. \quad (71)$$

Note that these projectors are not all mutually orthogonal and  $\rho_{AB}$  is a mixture with weight  $p_{nj} q_{nk}$  for all  $n, j, k$ . The density matrices

$$\rho_{AB}(n, j, k) = u_{nj} v_{nk} u_{nj}^\dagger v_{nk}^\dagger \quad (72)$$

(there is no sum) are not mutually orthogonal pure state projectors, except for  $n = 1$ . So, once the eigenvectors (eigenrays) are given, we can construct the density matrix of  $AB$  as a convex sum of projectors. For  $n = 1$ , what does  $\rho_{AB}$  contain that is not contained in  $\rho_A \otimes \rho_B$ ? In this case, it is a set of phase differences between various eigenvectors that make up  $\rho_{AB}$ . There are  $\text{rank}(\rho_A) \times \text{rank}(\rho_B)$  phases and hence one less phase difference. All this is very similar to the purification of mixed states. We write

$$\rho_A = \sum_j p_j \Pi_j^A \quad \rho_B = \sum_k q_k \Pi_k^B \quad (73)$$

and hence by hypothesis

$$\rho_{AB} = \rho_A \otimes \rho_B = \sum_{jk} p_j q_k \Pi_j^A \otimes \Pi_k^B. \quad (74)$$

So, for purification, we adopt the ansatz given earlier (we omit the Kronecker product symbol)

$$\tilde{\rho}_{AB} = \sum_{jkj'k'} (p_j q_k p_{j'} q_{k'})^{1/2} \frac{\Pi_j^A \Pi_k^B \Pi_{AB} \Pi_{j'}^A \Pi_{k'}^B}{\sqrt{\text{Tr}(\Pi_j^A \Pi_k^B \Pi_{AB} \Pi_{j'}^A \Pi_{k'}^B \Pi_{AB})}}. \quad (75)$$

This  $\tilde{\rho}_{AB}$  is a pure matrix with probability weights  $p_j q_k$  for the projectors  $\Pi_j^A \Pi_k^B$  and the suitable phase differences which number  $\text{rank}(\rho_A) \times \text{rank}(\rho_B) - 1$ . It is essential to choose  $\Pi_{AB}$  such that  $\text{Tr}(\Pi_{AB} \Pi_j^A \Pi_k^B) \neq 0$  for all  $j, k$ . But the density matrix so constructed will *not* lead to  $\rho_A$  and  $\rho_B$  as partial traces.

For a more general case of  $n \geq 2$ , we have the problem of re-diagonalizing

$$\rho_A = \sum_{n,j} p_{nj} \Pi_{nj}^A \quad \rho_B = \sum_{n,k} q_{nk} \Pi_{nk}^B. \quad (76)$$

Once this is done, we proceed as in the simple separate case ( $n = 1$ ) discussed above.

#### 14. Entanglement and straddling of fiducial projectors

The fiducial projector in the purification protocol generates the relative phases between the (two or more) density matrices for pure states which have been ‘superposed’. Such an operator is a ‘phase correlator’. The Hermitian fiducial projector and the projectors which are being superposed are all Hermitian, yet relative phases are introduced. Given two or more Hermitian matrices one can generate the Bargman phase from their product. For this to occur,

the operators cannot all commute (phase  $(ABC) \neq 0$ ). Simple examples may be provided by the Pauli matrices (or Dirac matrices), in our case, the phase difference between amplitudes is generated by the overlap of the fiducial matrix with the respective density matrices. Since we do a normalization in (12) only the phase of the overlap survives. There is a source of the phase interference introduced in our composition law (12). The question naturally arises—can we choose the fiducial projector  $P_0$  so as to produce any set of phase differences? The answer is affirmative but not unique.

The fiducial projector must straddle the pure states which are added, that is, it must have nonzero overlap with each of them. (It may or may not have overlap with other states.) Following up on this notion we find that the fiducial projector which restores a fully entangled pure state of a composite system straddles the eigenprojectors of the individual rays which are direct products of pure density matrices. (This is for separable systems, otherwise we get some direct product pure states and some fully entangled pure states.) This straddling implies that automatically the fiducial projector is a ‘nonlocal’ operator acting coherently on the subsystems. Since it is also a projector, it follows that this projector is a fully entangled pure state. Only such an entangled projector can restore full entanglement. Fully entangled operators can be multiplied by each other or added together to obtain fully entangled operators, but they will not be projectors of rank 1. This entanglement (and phase coherence) can be inherent in operators as well as in states.

## 15. Conclusions

We presented an intrinsic approach to different quantum phenomena which are entanglement and interference. The approach is intrinsic because it points out the unique basis for both phenomena which is the superposition principle of quantum states. But to use this superposition principle in the generic case of mixed states, one needs the addition formula for density operators. The discussed measure of entanglement is intrinsically connected with the given state of a composite system because it is determined completely by partial traces of the state density operator and by the deviation of the density operator from the tensor product of the partial traces. Thus, because the entanglement is the property related to the state superpositions (expressed in terms of a new addition rule of density operators using a fiducial projector) the fiducial projector becomes a useful tool for treating both phenomena—interference and entanglement.

To conclude, we point out new results of the paper.

The nonlinear addition rule for impure density matrices, which results in the pure density matrix given in (12) and (14), is a new purification procedure. The addition rule formula (25) for symbols of density operators of any kind (including Wigner distribution, tomograms, etc) is another new result of our consideration.

The notion of the measure of entanglement of arbitrary order for bipartite and multipartite systems for an arbitrary observable given by (41) is a new aspect of entanglement suggested in our study. As a partial case, the measure introduced contains the description of measure of entanglement of a density operator for bipartite system given in (40). The measure of entanglement is related directly to the intrinsic properties of the density operator of a composite system.

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### Appendix 1. Partial density matrices for the pure state of a composite system

Let us consider the pure state of a composite system which has two subsystems  $A$  and  $B$ . The pure state is described by a vector of the form

$$\psi = \sum_{i=1}^N \sum_{\alpha=1}^M C_{i\alpha} \varphi_i \chi_\alpha \quad (77)$$

where  $N$  is the dimension of the subsystem  $A$ ,  $M$  is the dimension of the subsystem  $B$ , and the orthogonal vectors  $\varphi_i$  ( $i = 1, \dots, N$ ) and  $\chi_\alpha$  ( $\alpha = 1, \dots, M$ ) form a basis in Hilbert spaces of the subsystem states.

The density operator of the pure state which corresponds to the decomposition (77) of the state vector has the form

$$\rho_{AB} = \psi \psi^\dagger = \sum_{i,j=1}^N \sum_{\alpha,\beta=1}^M C_{i\alpha} C_{j\beta}^* \varphi_i \varphi_j^\dagger \chi_\alpha \chi_\beta^\dagger. \quad (78)$$

The density matrix of the  $A$ -subsystem state in the chosen basis has the matrix elements expressed in terms of decomposition coefficients

$$(\rho_A)_{ij} = \sum_{\alpha=1}^M C_{i\alpha} C_{j\alpha}^*. \quad (79)$$

The density matrix of the  $B$ -subsystem state in the chosen basis has the matrix elements

$$(\rho_B)_{\alpha\beta} = \sum_{i=1}^N C_{i\alpha} C_{i\beta}^*. \quad (80)$$

Both density matrices  $\rho_A$  and  $\rho_B$  are nonnegative Hermitian matrices and  $\text{Tr} \rho_A = \text{Tr} \rho_B = 1$ . Let us calculate the parameters

$$\mu_n^{(A)} = \text{Tr}(\rho_A)^n \quad \mu_n^{(B)} = \text{Tr}(\rho_B)^n \quad (81)$$

for arbitrary integer  $n$ .

One can easily see that

$$\mu_n^{(A)} = \mu_n^{(B)}. \quad (82)$$

In fact,

$$\mu_n^{(A)} = \sum_{i_1, i_2, \dots, i_n=1}^N \sum_{\alpha_1, \alpha_2, \dots, \alpha_n=1}^M C_{i_1 \alpha_1} C_{i_2 \alpha_1}^* C_{i_2 \alpha_2} C_{i_3 \alpha_2}^* \cdots C_{i_{n-1} \alpha_{n-1}} C_{i_n \alpha_{n-1}}^* C_{i_n \alpha_n} C_{i_1 \alpha_n}^* \quad (83)$$

and

$$\mu_n^{(B)} = \sum_{i_1, i_2, \dots, i_n=1}^N \sum_{\alpha_1, \alpha_2, \dots, \alpha_n=1}^M C_{i_1 \alpha_1} C_{i_1 \alpha_2}^* C_{i_2 \alpha_2} C_{i_2 \alpha_3}^* \cdots C_{i_{n-1} \alpha_{n-1}} C_{i_{n-1} \alpha_n}^* C_{i_n \alpha_n} C_{i_n \alpha_1}^*. \quad (84)$$

The terms without a star are the same in both expressions (83) and (84). These terms are invariant with respect to arbitrary permutations

$$1, 2, \dots, n \rightarrow s_1, s_2, \dots, s_n.$$

The terms with a star look different in (83) and (84), but since both sums (83) and (84) are invariant with respect to arbitrary permutations, let us make the particular permutation

$$1, 2, \dots, n-1, n \rightarrow 2, 3, \dots, n, 1$$

in sum (84). The terms without a star are invariant and the terms with a star in (84) after permutation coincide with the terms with a star in (83). This proves equality (82) which means that the eigenvalues and rank of the matrices  $\rho_A$  and  $\rho_B$  are the same. It is clear that the proof can be extended to multipartite composite system  $AB \dots C$ . Thus we get the following result. Given a pure state of a multipartite quantum system  $\rho_{AB \dots C}$ , the eigenvalues and ranks of the density matrices  $\rho_A, \rho_B, \dots, \rho_C$  are equal.

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